REAL INVERSION AND JUMP FORMULA FOR THE LAPLACE TRANSFORM, PART II*

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ABSTRACT

A generalization of the Hirschman inversion formula and a new jump formula for the Laplace transform are proved.

In this part of our paper we shall make use of theorems and methods of the first part and a generalization of formula (25) p. 132 in [1], to get a new jump formula and a generalization of the known Hirschman inversion formula for the Laplace transform

§5. **Real inversion formula of Hirschman type.** We begin by defining for $y > 0$ and the real sequences $\{g(k)\}$ and $\{a_k\}$ $(k \geqq 1)$, the operator $W[k, y, g(k)a_k; f]$ operating on the function $f(x)$ by

$$(1.5) \qquad W[k, y, g(k), a_k; f] = \frac{(y(g(k) + a_k))^{g(k)\cdot 1/2}}{\Gamma(g(k))} \cdot$$

$$\cdot \int_0^\infty u^{1/2g(k)} J_{g(k)}(2uy(g(k) + a_k)^{1/2}) f(u) du$$

where $J_\nu(z)$ is the Bessel function of the first kind, of order ν and $\Gamma(z)$ is the Γ-function.

A function $\phi(t)$ belongs to class G if $\phi(t) \in L_1(0,R)$, $R > 0$ and, for $\theta(t) = \int_0^t \phi(u)du$, we have $\theta(t) = 0(t^r)$ $(t \uparrow \infty)$ for some finite r and $\theta(t) = O(t^m)(t \uparrow 0)$ for each $m > 0$.

We say that $\{g(k)\} \in D$ if $\{g(k)\}$ is a real sequence satisfying $g(k) \sim k(k \uparrow \infty)$.

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(*) This paper is the second part of a paper with the same name (see this Journal v., 1, 1963, pp. 85–104). The notations used here are the same as in the first part. §1 – §4 belong to the first part.

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THEOREM 1.5. *Suppose that $f(x)$ is the Laplace transform of $\phi(t)$, $\phi(t) \in G$ and let $\{g(k)\} \in D$. Then, for any fixed $y > 0$, we have:*

(i) *If $\{a_k\} \in A(\lambda)$ and both $\phi(y \pm 0)$ exist, then* $\lim_{k\to\infty} W[k,y,g(k),a_k;f] = (1 - N(\lambda)\phi(y-0) + N(\lambda)\phi(y+0)$

(ii) *If $\{a_k\} \in B^+$ and $\phi(y+)$ exists, then* $\lim_{k\to\infty} W[k,y,g(k),a_k;f] = \phi(y+)$.

(iii) *If $\{a_k\} \in B^-$ and $\phi(y-)$ exists, then* $\lim_{k\to\infty} W[k,y,g(k)a_k;f] = \phi(y-)$.

(iv) *If $\{a_k\} \in B$ and y is a point of continuity of $\phi(t)$, then* $\lim_{k\to\infty} W[k,y,g(k)a_k;f] = \phi(y)$.

(v) *If $\{a_k\} \in A^*$, both $\phi(y \pm 0)$ exist and are equal, then* $\lim_{k\to\infty} W[k,y,g(k),a_k;f] = \phi(y+0)(= \phi(y-0))$.

Theorem 1.5 with $g(k) = k$ and $a_k = 0$ is Hirschman's inversion formula([2]).

A function $\alpha(t)$ belongs to class H if for some finite $r > 0$ $\alpha(t) = 0(t^r)(t \uparrow \infty)$ and for each $m > 0$ $\alpha(t) = 0(t^m)(t \downarrow 0)$.

THEOREM 2.5. *Let $f(x)$ be the Laplace-Stieltjes transform of a function $\alpha(t) \in H$ and let $\{g(k)\} \in D$. Then for any $y > 0$, in the following four cases*

$$\lim_{k\to\infty} \int_0^y W[k,u,g(k),a_k;f]\,du \tag{2.5}$$

and

$$\lim_{k\to\infty} W[k,y,g(k),a_k;f_1] \text{ (where } f_1(x) = f(x)/x) \tag{3.5}$$

exist and are both equal (respectively) to

$$\begin{cases} (1-N(\lambda))\alpha(y-) + N(\lambda)\alpha(y+) & \text{if } \{a_k\} \in A(\lambda) \\ \alpha(y+) & \text{if } \{a_k\} \in B^+ \\ \alpha(y-) & \text{if } \{a_k\} \in B^- \\ \alpha(y) & \text{if } \alpha(y-) = \alpha(y+) \text{ and } \{a_k\} \in B. \end{cases} \tag{4.5}$$

In the special case $g(k) = k$ and $a_k = 0$, Theorem 2.5 is Theorem 4a of [2]. By the heuristic method of §2 we get some new jump formulae. We state here and prove in §7 the final results, without the heuristic calculation.

THEOREM 3.5. *Let $\{g(k)\} \in D$ and $\{a_k\} \in A(\lambda)$ for some real λ. Suppose $f(x)$ is the Laplace transform of $\phi(t) \in G$. If for some $y > 0$ both $\phi(y \pm 0)$ exist, then*

$$\lim_{k\to\infty} e^{\lambda^2/2} \frac{\sqrt{2\pi k}}{\Gamma(g(k))} (y(g(k)+a_k))^{(g(k)-1)/2} \cdot \int_0^\infty x^{(g(k)+1)/2} J_{g(k)-1}(2\{xy(g(k)+a_k)\}^{1/2}) \cdot f(x)dx = \phi(y+0) - \phi(y-0).$$

THEOREM 4.5. *Let $\{g(k)\} \in D$ and $\{a_k\} \in A(\lambda)$. Suppose that $f(x)$ is the Laplace Stieltjes transform of $\alpha(t) \in H$. Then for any fixed $y > 0$*

(a)
$$\lim_{k\to\infty} e^{\lambda^2/2} \frac{\sqrt{2\pi k}(g(k)+a_k)^{(g(k)-1)/2}}{\Gamma(g(k))} \cdot$$
$$\cdot \int_0^y du \int_0^\infty (xu)^{(g(k)+1)/2} J_{g(k)-1}(2\{xu(g(k)+a_k)\}^{1/2}).$$
$$\cdot f(x)dx = \alpha(y+) - \alpha(y-).$$

(b)
$$\lim_{k\to\infty} e^{\lambda^2/2} \frac{\sqrt{2\pi k}(y(g(k)+a_k))^{(g(k)-1)/2} y}{\Gamma(g(k))} \int_0^\infty x^{(g(k)-1)/2} f(x) \cdot$$
$$\cdot J_{g(k)-1}(2(xy(g(k)+a_k))^{1/2})dx = \alpha(y+) - \alpha(y-).$$

Theorems 3.5 and 4.5 for $g(k) = k$ and $a_k = 0$ ($k \geqq 1$) yield simpler results. For instance, from Theorem 3.5 we get

Corollary 1.5 *Under the assumption of theorem* 3.5 *we have*

$$\lim_{k\to\infty} \frac{\sqrt{\pi 2}\, y^{(k+1)/2}}{(k-1)!} k^{k/2} \int_0^\infty x^{(k+1)/2} J_{k-1}(2(xyk)^{1/2}) f(x)dx$$
$$= \phi(y+0) - \phi(y-0).$$

§6. **A general formula for the Laplace transform.** In this section we prove a generalization of formula (2.5) on page 132 of [1]. This result will be used in proving the theorems of §5.

LEMMA 1.6. *Let $f(x)$ be the Laplace transform of $\phi(u)$. Define $\theta(t) = \int_0^t \phi(u)du$ $t \geqq 0$. Suppose that, for the real number v and the two integers p, j satisfying*

(1.6)
$$p \equiv j (\text{mod } 2),\ p + j \geqq 0,\ v + \frac{p-j}{2} > -2,$$

we have

(2.6)
$$\theta(t) = 0(t^r)(t \uparrow \infty) \quad \textit{for some } r \geqq 0\ r < v + 1 + \frac{p-j}{2}$$

and

(3.6)
$$\theta(t) = 0(t^m)(t \downarrow 0) \qquad \textit{for some } m > \frac{v+p}{2} + \frac{3}{4}.$$

Then, for $s > 0$,

(4.6)
$$S^{\frac{j-v}{2}} \int_0^\infty u^{(v+p)/2} J_{v-j}(2\sqrt{us}) f(u)du$$
$$= (-1)^{(p+j)/2} \int_0^\infty \phi(t) \left(\frac{d}{dt}\right)^{(p+j)/2} \{e^{-s/t} t^{-v+j-1}\} dt$$
$$= \int_0^\infty \phi\left(\frac{1}{t}\right) \left(t^2 \frac{d}{dt}\right)^{(p+j)/2} \left\{e^{-st} t^{v-j+1}\right\} \frac{dt}{t^2}.$$

The special case $p = j = 0$ of Lemma 1.6 is formula (2.5) on page 132 of [1].

Proof. We have for $u > 0$,

$$f(u) = \int_0^\infty e^{-ut}\phi(t)dt = u \int_0^\infty e^{-ut}\theta(t)dt$$

where $f_1(u) = u \int_0^\infty e^{-ut}|\theta(t)|dt$ exists for $u > 0$. Now (see Widder [3] p. 181 Theorem 1.) $f_1(u) = 0(u^{-r})(u \downarrow 0)$ for the r of (2.6), $f_1(u) = 0(u^{-m})(u \uparrow \infty)$ for the m of (3.6), and as is well known,

$$J_\alpha(u) = 0(u^\alpha)(u \downarrow 0), \qquad J_\alpha(u) = 0(u^{-1/2})(u \uparrow \infty)$$

for real α. Therefore

$$\text{(5.6)} \qquad s^{(j-\nu)/2} \int_0^\infty u^{(\nu+p)/2} |J_{\nu-j}(2\sqrt{us})/u \int_0^\infty e^{-ut} |\theta(t)| dt\, du$$

converges for $s > 0$; now

$$\text{(6.6)} \quad I = s^{(j-\nu)/2} \int_0^\infty u^{(\nu+p)/2} J_{\nu-j}(2\sqrt{us}) f(u)\, du$$

$$= s^{(j-\nu)/2} \int_0^\infty u^{(\nu+p)/2} J_{\nu-j}(2\sqrt{us})\, u \int_0^\infty e^{-ut}\theta(t)dt\, du$$

(by (5.6) and Fubini's theorem)

$$= s^{(j-\nu)/2} \int_0^\infty \theta(t) \left\{ \int_0^\infty u^{(\nu+p)/2+1} J_{\nu-j}(2\sqrt{us}) e^{-ut} du \right\} dt.$$

We have

$$\text{(7.6)} \quad J = S^{(j-\nu)/2} \int_0^\infty u^{((\nu+p)/2)+1} J_{\nu-j}(2\sqrt{us})\, e^{-ut} du$$

$$= \int_0^\infty e^{-ut} \left\{ \sum_{k=0}^\infty (-1)^k \frac{S^k u^{k+\nu+((p-j)/2)+1}}{k!_0\Gamma(k+\nu-j+1)} du. \right.$$

Changing formally the order of summation and integration we get

$$\text{(8.6)} \quad J = \sum_{k=0}^\infty (-1)^k \frac{S_k}{k!\Gamma(k+\nu-j+1)} \cdot \int_0^\infty e^{-ut} u^{k+\nu+((p-j)/2)+1} du$$

$$\left(\nu + \frac{p-j}{2} + 1 > -1,\ \nu + \frac{p+j}{2} > -2\right)$$

$$= \sum_{k=0}^\infty (-1)^k \frac{\Gamma\left(k+\nu+\frac{p-j}{2}+2\right) s^k}{k!\Gamma(k+\nu-j+1)} \left(\frac{1}{t}\right)^{k+\nu+(p-j)/2+2}$$

$$= (-1)^{(p+j)/2} \left(\frac{d}{dt}\right)^{((p+j)/2)+1} \{t^{-\nu+j-1} e^{-s/t}\}.$$

Now Fubini's theorem and the fact that

$$\sum_{k=0}^{\infty} \frac{\Gamma\left(k+v+\left(\frac{p-j}{2}\right)+2\right)}{k!\Gamma(k+v-j+1} z^k$$

is an integral function justify the formal change of summation and integration. Combining (6.6) and (8.6) we get

$$I = (-1)^{(p+j/2)+1} \int_0^\infty \theta(t)\left(\frac{d}{dt}\right)^{((p+j)/2)+1} \{t^{-v+j+1} e^{-s/t}\}dt$$

$$= (-1)^{(p+j)/2} \int_0^\infty \phi(t)\left(\frac{d}{dt}\right)^{(p+j/2)} \{t^{-v+j-1} e^{-s/t}\}dt \text{ (by integration by parts)}$$

$$= \int_0^\infty \phi\left(\frac{1}{u}\right)\left(u^2 \frac{d}{du}\right)^{(p+j)/2} \{e^{-su}u^{v-j+1}\} \frac{du}{u^2}\left(\text{where } u = \frac{1}{t}.\right)$$

Q.E.D.

§7. **Proofs of the Theorems of** §5.

Proof of Theorem 1.5. Under the suppositions of our Theorem, the substitutions (in Lemma 1.6) of $p = j = 0$, $s = y(g(k) + a_k)$ and $v = g(k)$ imply

$$(1.7)\ I_k = W[k, y, g(k), a_k; f(x)] = \frac{\{y(g(k)+a_k)\}^{g(k)}}{\Gamma(g(k))} \cdot \int_0^\infty \phi(t) t^{-g(k)-1} \cdot$$

$$\cdot\ e^{-(g(k)+a_k)y/t}dt = \frac{g(k)^{g(k)}}{\Gamma(g(k))}\left\{\int_0^{1-\delta} + \int_{1-\delta}^{g(k)/g(k)+a_k}\right.$$

$$\left. + \int_{g(k)/g(k)+a_k}^{1+\delta} + \int_{1+\delta}^\infty \right\} e^{-g(k)/z} z^{-g(k)-1} \phi\left(zy\left(1+\frac{a_k}{g(k)}\right)\right)dz$$

$$= I_{k,1} + I_{k,2} + I_{k,3} + I_{k,4} \text{ for some } \delta, \quad 0 < \delta < 1.$$

Define

$$(2.7) \qquad \alpha(t) = \int_0^t \phi(zy(1 + a_k g(k)^{-1}))dz \qquad \text{for } t \geqq 0.$$

Now for any sequence $\{a_k\}$ satisfyiyng $a_k = 0(k)$ $(k\uparrow\infty)$ we have $\alpha(t) = 0(t^r)$, $(t\uparrow\infty)$ for some finite r. Hence

$$I_{k,4} \sim \sqrt{\frac{g(k)}{2r}}e^{g(k)}\Big\{ [e^{-g(k)/z}z^{-g(k)-1}\alpha(z)]_{1+\delta}^{\infty} +$$

$$+ \int_{1+\delta}^{\infty} e^{-g(k)/z}z^{-g(k)-1}\left(\frac{g(k)}{z^2}+\frac{g(k)+1}{z}\right)\alpha(z)dz\Big\} \to 0 \text{ as } (k\uparrow\infty)$$

since $0 < e\cdot e^{-z}\cdot z^{-1} < 1$ for $z > 1+\delta$. Similarly

$$\lim_{k\to\infty} I_{k,1} = 0$$

The calculations in the proof up to now depend only on $a_k = 0(k)$ $(k\uparrow\infty)$. This condition is satisfied in all cases (i)–(v). Now we prove the case (i) of our theorem. We estimate $I_{k,3}$ by means of Theorem 2.3. Substitute there $a = 1$, $b = 1+\delta$, $h(z) = -(1/z) - \log(z)$, our $-\lambda$ for the λ there and for the $\{a_k\}$ there, substitute $\{(a_k g(k)/g(k) + a_k)\}$ $(k \geqq 1)$ where $\{a_k\}$ is the sequence given in our theorem. It is easy to see that the last sequence belongs to $A(-\lambda)$. Also choose

$$\phi_k(u) = u^{-1}\phi(uy(1 + a_k\cdot g(k)^{-1})).$$

Theorem 2.3 yields for these substitutions

$$\lim_{k\to\infty} I_{k,3} = \phi(y+0)(1-N(-\lambda)) = N(\lambda)\phi(y+0).$$

Similarly we get by using Corollary 2.3 instead of Theorem 2.3.

$$\lim_{k\to\infty} I_{k,2} = \phi(y-0)N(-\lambda) = (1-N(\lambda))\phi(y-0).$$

Combining the estimations for $I_{k,i}(i = 1,2,3,4)$ we obtain the proof of case (i). The proofs of conclusions (ii)–(v) are similar to the proofs of (ii)–(v) in Theorem 1.2.

Proof of Theorem 3.5. Choose, in Lemma 1.6, $p = j = 1$, $v = g(k)$ and $s = y(g(k) + a_k)$. Then $\phi(t)$ which satisfies the asumptions of Theorem 3.5, satisfies for $k \geqq k_0$ the asumptions of Lemaa 1.6 for the above choice of the parameters. Hence, by Lemma 1.6,

$$e^{\lambda^2/2}\frac{\sqrt{2\pi k y}}{\Gamma(g(k))}\{y(g(k)+a_k)\}^{(g(k)-1)/2}\cdot$$

$$\cdot\int_0^\infty x^{(g(k)+1)\cdot 1/2}J_{g(k)-1}(2\{xy(g(k)+a_k)\}^{1/2})f(x)dx$$

$$=-e^{\lambda^2/2}\frac{\sqrt{2\pi k}\,y}{\Gamma(g(k)}\{y(g(k)+a_k)\}^{g(k)-1}\cdot\int_0^\infty \phi(t)e^{-y(g(k)+a_k)/t}e^{-g(k)-2}\cdot$$

$$\cdot\{(g(k)+a_k)y-tg(k)\}dt=$$

$\left(\text{and by the substitution } z=\dfrac{tg(k)}{g(k)+a_k}\right)$

$$=-e^{\lambda^2/2}\frac{\sqrt{2\pi k}}{\Gamma(g(k))}g(k)^{g(k)}\ \frac{g(k)}{g(k)+a_k}\left\{\int_0^{1-\delta}+\int_{1-\delta}^{g(k)/(g(k)+a_k)}+\right.$$

$$\left.+\int_{g(k)(g(k)+a)}^{1+\delta}+\int_{1+\delta}^\infty\right\}\phi(zy(g(k)+a_k)g(k)^{-1})e^{-g(k)/z}z^{g(k)-2}(1-z)dz$$

$$=I_{k,1}+I_{k,2}+I_{k,3}+I_{k,4}\text{ where } 0<\delta<1.$$

We estimate $I_{k,3}$ by means of Theorem 4.3. Substitute there $a=1$, $b=1+\delta$, $h(z)=-z^{-1}-\log z$, our $-\lambda$ for the λ there, and for $\{a_k\}$ there, substitute $\{-(a_kg(k)/g(k)+a_k)\}(k\geqq 1)$ where $\{a_k\}$ is the sequence given in our theorem. It is easy to see that the last sequence belongs to $A(-\lambda)$. Also choose $\phi_k(u)=u^{-2}\phi(uy(1+a_{k/}g^{(k)}))$. Theorem 4.3. yields for these substitutions

$$\lim_{k\to\infty}I_{k,3}=\phi(y+0) \tag{3.7}$$

The above substitution and the argument of the proof of Theorem 1.5 yield

$$\lim_{k\to\infty}I_{k,2}=-\phi(y-0)\quad\text{and}\quad\lim_{k\to\infty}I_{k,1}=\lim_{k\to\infty}I_{k,2}=0$$

Proof of Theorems 2.5 and 4.5. The arguments used in proving Theorems 1.5 and 3.5 and some simple modifications prove our theorems. Q.E.D.

§8. REMARKS.

The arguments used in proving Theorems 1.5 and 3.5 yield the following result too.

THEOREM 1.8. *Let a be real and let $\{g(k)\}\in D$. Suppose that $f(x)$ is the Laplace transform of $\phi(t)$, $\int_0^t e^{-au}\phi(u)du=O(t^r)$ $(t\uparrow\infty)$ for some finite r; then for any fixed $y>0$ and any $\eta>0$*

(i) $\lim_{k\to\infty} e^{(\eta+y)a} W[k, y+\eta, g(k), a_k; f_2] = (1-N(\lambda))\phi(y-0) + N(\lambda)\phi(y+0)$

(ii) $\lim_{k\to\infty} (e^{\lambda^2/2}\sqrt{2\pi k}(y+\eta)\, e^{a(y+\eta)} \{(y+\eta)(g(k)+a_k)\}^{(g(k)-1)/2}/\Gamma(k)) \cdot \int_0^\infty x^{g(k)+1/2} \cdot$

$I_{g(k)-1}(2\{x(y+\eta)(g(k)+a_k)\}^{1/2}) f_2(x)dx = \phi(y+0) - \phi(y-0).$

where $f_2(x) = e^{-\eta(x+a)} f(x+a)$.

Results analogous to (ii)–(v) of Theorem 1.5 can also be stated in the present set up.

In the same way that Theorem 1.8 follows from Theorems 1.5 and 3.5 it is possible to obtain analogous results from Theorems 2.5 and 4.5 by replacing the assumption $\alpha(t) \in H$ by $\alpha(t) = 0(t^r z^{at})$ $(t \uparrow \infty)$ (for some real a and r).

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